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Area distribution of an elastic Brownian motion

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Abstract

We calculate the excursion and meander area distributions of the elastic Brownian motion by using the self-adjoint extension of the Hamiltonian of the free quantum particle on the half line. We also give some comments on the area of the Brownian motion bridge on the real line with the origin removed. We will focus on the power of self-adjoint extension to investigate different possible boundary conditions for the stochastic processes. We also discuss some possible physical applications.

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1. Introduction

In this paper, we study some area distributions of an elastic Brownian motion [1], such as the elastic Brownian excursion area and the elastic Brownian meander area. The goal apart from the calculation of the area distribution of the elastic Brownian motion is to give a unified framework to study different area distributions for the Brownian motion with generic boundary conditions. This generalization is in close connection with the concept of the self-adjoint extension in quantum mechanics. The self-adjoint extension gives a reasonable classification for the possible boundary conditions of the quantum particle and so for the stochastic process.

Different area distributions of the Brownian motion in one dimension were calculated by the mathematicians in the last century. The Brownian motion area was first calculated by Kac [2]. The Brownian excursion area was calculated by Darling [3] and Louchard [4] and the Brownian meander area was calculated by Perman and Wellner [5]. For the applications and other distributions of the areas see [6, 7] and also the complete review by Janson [8], and references therein.

The quantum mechanics methods first used by Comtet and Majumdar [9] to re-drive the different area distributions of the Brownian motion. This method which is more common in the physics literature apart from simplicity can give a unique way to calculate the different area distributions. It is also very useful to generalize the results to more complicated stochastic processes. The self-adjoint extension of the Hamiltonian operator gives us the variety of different possible boundary conditions in the presence of the boundary [10]. In this paper, we will focus on the area distribution of the elastic Brownian motion which is equivalent to the free quantum particle on the half line. We also calculate the area distribution of the stochastic process equivalent to the free particle on the real line with a hole at the origin. Of course the area distribution is just one of the myriad of possible distributions that one can investigate for the elastic Brownian motion and the Brownian motion on the pointed real line. We will summarize some of these distributions in the last section.

The paper is organized as follows. In the second section, we define the elastic Brownian motion and its connection to the quantum particle on the half line. In the third section, we use the same method as [9] to calculate the area distribution of the elastic Brownian excursion and meander. Different limits of our calculation will give the well-known results. We will conclude this section with some immediate application of our results and giving some hints about other useful Brownian functionals. In the fourth section, using the results of section 3 we will give the distribution of the area for Brownian bridge with a point defect at the origin. Finally, in the last section, we will summarize our results and possible future directions.

2. Elastic Brownian motion and quantum mechanics

The elastic Brownian motion is the natural generalization of the Brownian motion in the half line with special boundary condition on the origin. To define the process we need first to introduce the local time. The definition of the local time of the path ω at the point a is as follows:

$$t_l(a) := \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\int_0^T \mathbf{1}_{x+\varepsilon}(B_s)\,\mathrm{d}s, \tag{2.1}$$

where $\mathbf{1}_{x+\varepsilon}(B_s)$ is the indicator for the time that the process is in the interval $[0, x+\epsilon]$. One can naively write the above equation as an integral over a delta function as $t_l(\omega, 0) = \int_0^T \delta(x(t))\,\mathrm{d}t$. The local time has dimensions of the inverse of velocity and can be written as $t_l(a) = n\frac{\Delta t}{\Delta x}$ for the discrete random walk, where n is the number of times the path hits the origin. Then one can write the exponential of the local time as

$$\exp\left(\frac{-2\pi}{\eta}t_l\right) \approx \left(1 - \frac{2\pi}{\eta}\Delta x\right)^n. \tag{2.2}$$

To go to the discrete level we multiplied t_l with $\frac{(\Delta x)^2}{\Delta t}$ that comes from the central limit theorem. Equation (2.2) is derived by considering $1 - \frac{2\pi}{\eta}t_l$ as the probability for a single reflection from the origin it is possible to interpret the exponential of the local time as the probability that a particle on a given path is reflected from the origin[1]. Then the Green's function of the elastic Brownian motion is just by the following expectation value:

$$G(x, y, T) = \left\langle \exp\left(\frac{-2\pi}{\eta}t_l\right)\right\rangle. \tag{2.3}$$

The elastic Brownian motion is in close connection with the quantum particle on the half line. The Hamiltonian operator of the quantum particle on the half line has self-adjoint extension with the following boundary condition:

$$\psi(0) = \frac{-\eta}{2\pi}\frac{\mathrm{d}\psi}{\mathrm{d}x}\bigg|_{x=0}. \tag{2.4}$$

[1] The name elastic Brownian motion comes from this property of the process. $\eta = \infty$ is the reflecting barrier, the particle will be reflected from the boundary with probability 1. $\eta = 0$ is the absorbing barrier, the particle will be absorbed after hitting the boundary. Other cases between two extreme cases are called elastic barrier.

The above boundary condition is called Robin boundary condition. The energy of the particle is $E_k = \frac{1}{2}k^2$ and the wavefunctions are

$$\psi_k = \sqrt{\frac{2}{\pi}}\cos(kx + \delta_k), \tag{2.5}$$

where $\tan(\delta_k) = \frac{2\pi}{\eta k}$ and δ_k is the phase shift corresponding to the s-wave. The solutions are normalizable and complete.

The Hamiltonian is self-adjoint for all of the real values of η but to avoid the cases with bound states we will consider just non-positive η. $\eta = 0$ is the Dirichlet boundary condition and $\eta \to -\infty$ is the Neumann boundary condition. The Green's function with respect to the solutions of the Hamiltonian has the following form:

$$G_\eta(x, y, t) = \int_0^\infty \mathrm{d}k\ \mathrm{e}^{-\mathrm{i}E_k t}\varphi(y)\varphi^*(x). \tag{2.6}$$

Using the above equation, the Green's function has the following form for arbitrary values of the self-adjoint extension [11–13]:

$$G_\eta(x, y, t) = G_F(x-y, t) + G_F(x+y, t) + \frac{4\pi}{\eta}\int_0^\infty \mathrm{d}w\ \mathrm{e}^{\frac{2\pi}{\eta}w} G_F(x+y+w) \qquad \eta \leqslant 0; \tag{2.7}$$

$$G_\eta(x, y, t) = G_F(x-y, t) + G_F(x+y, t) - \frac{4\pi}{\eta}\int_0^\infty \mathrm{d}w\ \mathrm{e}^{\frac{-2\pi}{\eta}w} G_F(x+y-w) + \frac{4\pi}{\eta}\mathrm{e}^{\mathrm{i}\frac{2\pi^2 \mathrm{t}}{\eta^2}}\mathrm{e}^{-\frac{2\pi}{\eta}(\mathrm{x+y})} \qquad \eta \geqslant 0, \tag{2.8}$$

$$G_F(x-y, t) = \frac{1}{\sqrt{2\pi i t}}\mathrm{e}^{\mathrm{i(x-y)^2/2t}}. \tag{2.9}$$

For the special cases, Dirichlet and Neumann, the results are as follows:

$$G_{\eta=0}(x, y, t) = G_F(x-y, t) - G_F(x+y, t), \tag{2.10}$$

$$G_{\eta\to-\infty}(x, y, t) = G_F(x-y, t) + G_F(x+y, t). \tag{2.11}$$

One can use the above equations to get the Green's function of the elastic Brownian motion by just Wick rotation. The important point of this section is the possibility of using quantum mechanics language to describe the elastic Brownian motion. The other interesting point is the possibility of extending this equality into the level of the path integral representation [11–13]. In the following section, we use this correspondence to calculate different area distributions of the elastic Brownian motion.

3. Area of the elastic Brownian motion

In this section, we will solve the problem of the area for the restricted Brownian motion, in particular, we will solve the problem of the area distribution of the elastic Brownian excursion and elastic Brownian meander. In the extreme limits we will get the well-known results.

3.1. The area under the elastic Brownian excursion

In this section, we calculate excursion area of the elastic Brownian motion. The definition of the excursion area is as follows: take an elastic Brownian process that starts at $x(0) = \epsilon$ and returns after time T to $x(T) = \epsilon$, without crossing the origin in between. We are interested in the probability density $P(A, T, \epsilon)$ of the area $A = \int_0^T x(\tau)\,\mathrm{d}\tau$ for a fixed ϵ and then finally take the limit $\epsilon \to 0$, it plays the role of the regulator and can be treated independent of the limit in the local time process. This regularization is just necessary for the Dirichlet boundary condition but we are happy to keep it in our calculation to see its relevance in the calculation of the Dirichlet boundary condition. To do this calculation we map the problem to the quantum-mechanical problem. Now we use the method of Comtet and Majumdar [9] to calculate the excursion area distribution. To satisfy the constraint that process stays non-negative between 0 and T one can multiply the above measure with the indicator function $\prod_{\tau=0}^{T}\theta[x(\tau)]\exp\left(\frac{-2\pi}{\eta}t_l\right)$ which θ is the step function. The distribution $P(A,T)$ of the area under the elastic Brownian excursion can then be expressed as the following quantum-mechanical problem:

$$P^{\eta}(A,T)=\frac{1}{Z_E^{\eta}}\int_{x(0)=\epsilon}^{x(\tau)=\epsilon}\mathcal{D}x(\tau)\,\mathrm{e}^{-\int_0^T\left(\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2+\frac{2\pi}{\eta}\delta(x(t))\right)}\prod_{\tau=0}^{T}\theta[x(\tau)]\delta\left(\int_0^T x(\tau)\mathrm{d}\tau-A\right),\quad (3.1)$$

where Z_E^{η} is the normalization and corresponds to the quantum mechanics of a particle with infinite wall at the origin and zero potential at the positive real line

$$Z_E^{\eta}=\langle\epsilon\mid\mathrm{e}^{-H_0T}\mid\epsilon\rangle.\quad (3.2)$$

The Hamiltonian, H_0 is equal to the self-adjoint Hamiltonian that we discussed in the previous section. After integration, for small ϵ we have

$$Z_E^{\eta}\simeq 2(\eta-2\pi\epsilon)^2\left(\frac{1}{\sqrt{2\pi T}\eta^2}-\frac{\pi}{\eta^3}\mathrm{e}^{\frac{2\pi^2T}{\eta^2}}\,\mathrm{Erfc}\left(\sqrt{2T}\frac{\pi}{\eta}\right)\right).\quad (3.3)$$

The integral for the Dirichlet and Neumann cases is

$$Z_E^{0}\simeq\frac{1}{\sqrt{2\pi}}\epsilon^2T^{-3/2}+O(\epsilon^3),\qquad \eta=0,$$

$$Z_E^{\infty}\simeq\sqrt{\frac{2}{\pi T}}+O(\epsilon^2),\qquad \eta\to-\infty.\quad (3.4)$$

The above partition functions are just the probability that an elastic Brownian motion goes from $x(0)=\epsilon$ to $x(\tau)=\epsilon$ in time T without crossing the origin. Taking the Laplace transform $P(\lambda,T)=\int_0^{\infty}P(A,T)\mathrm{e}^{-\lambda A}\mathrm{d}A$ of both sides of equation (3.1) gives

$$P^{\eta}(\lambda,T)=\frac{1}{Z_E}\int_{x(0)=\epsilon}^{x(\tau)=\epsilon}\mathcal{D}x(\tau)\,\mathrm{e}^{-\int_0^T\left(\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2+\frac{2\pi}{\eta}\delta(x(t))+\lambda x(\tau)\right)}\prod_{\tau=0}^{T}\theta[x(\tau)].\quad (3.5)$$

In the numerator, we have the propagator $\langle\epsilon\mid\mathrm{e}^{-H_1T}\mid\epsilon\rangle$ where $H_1=-\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2+V(x)$ and $V(x)=\lambda x$ for $x>0$ and infinite for $x\leqslant 0$. We absorb the Dirac delta function into the boundary condition as the case of the quantum particle on the half line, in the other words, we consider the self-adjoint extension of this operator. The boundary condition of the wavefunction after self-adjoint extension is the same as (2.4). The solution to the Schrödinger equation is given by the Airy function as follows:

$$\psi_i^{\eta}(x)=\frac{\mathrm{Ai}((2\lambda)^{1/3}(x-E/\lambda))}{\sqrt{\int_0^{\infty}\mathrm{Ai}^2((2\lambda)^{1/3}(y-E/\lambda))\,\mathrm{d}y}}.\quad (3.6)$$

Using the boundary conditions one can determine the discrete eigenvalues as follows:

$$\mathrm{e}_i^{\eta}=2^{-1/3}\lambda^{2/3}c_i^{\eta},\qquad \frac{\mathrm{Ai}\left(-c_i^{\eta}\right)}{\mathrm{Ai}'\left(-c_i^{\eta}\right)}=-\frac{\eta(2\lambda)^{1/3}}{2\pi}.\quad (3.7)$$

Unfortunately, since the above equation is transcendental the exact form of c_i^{η} for the generic boundary condition is not available. However, for the Dirichlet and Neuman boundary conditions the solutions are just the magnitude of the zeros of $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ on the negative real axes, respectively, we show them by $-c_i^0$ and $-c_i^\infty$. The first few real roots of $\mathrm{Ai}(z)$ are approximately −2.338 11, −4.087 95, −5.520 56, −6.786 71, etc. The first few real roots of $\mathrm{Ai}'(z)$ are approximately −1.018 792, −3.248 197, −4.820 099, −6.163 307, etc. Then the wavefunctions in these two cases are

$$\psi_i^0(x) = (2\lambda)^{1/6}\frac{\mathrm{Ai}\big((2\lambda)^{1/3}x - c_i^0\big)}{\mathrm{Ai}'\big(-c_i^0\big)}, \tag{3.8}$$

$$\psi_i^\infty(x) = (2\lambda)^{1/6}\frac{\mathrm{Ai}\big((2\lambda)^{1/3}x - c_i^\infty\big)}{\sqrt{c_i^\infty}\mathrm{Ai}\big(-c_i^\infty\big)}. \tag{3.9}$$

To get the above results we used the identity $\int_x^\infty \mathrm{Ai}^2(x)\mathrm{d}x = -x\mathrm{Ai}^2(x) + \mathrm{Ai}'^2(x)$. Using the energy eigenvalues and wavefunctions one can write equation (3.5) as follows:

$$P^\eta(\lambda, T) = \frac{\langle\epsilon \mid \mathrm{e}^{-H_1 T} \mid \epsilon\rangle}{Z_E^\eta} = \frac{1}{Z_E^\eta}\sum_{i=1}^\infty |\psi(\epsilon)|^2 \mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^\eta T}. \tag{3.10}$$

For the Dirichlet and Neumann cases after considering small ϵ we have

$$P^0(\lambda, T) = \sqrt{2\pi}\lambda T^{3/2}\sum_{i=1}^\infty \mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^0 T}, \tag{3.11}$$

$$P^\infty(\lambda, T) = \sqrt{\frac{\pi T}{2}}(2\lambda)^{1/3}\sum_{i=1}^\infty \frac{1}{c_i^\infty}\mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^\infty T}. \tag{3.12}$$

One can write equations (3.11) and (3.12) in the scaling form as follows:

$$P^0(\lambda, T) = s\sqrt{2\pi}\sum_{i=1}^\infty \mathrm{e}^{-2^{-1/3}s^{2/3}c_i^0}, \tag{3.13}$$

$$P^\infty(\lambda, T) = u\sqrt{\frac{\pi}{2}}2^{1/3}\sum_{i=1}^\infty \frac{1}{c_i^\infty}\mathrm{e}^{-2^{-1/3}u^2 c_i^\infty}, \tag{3.14}$$

where $s = \lambda T^{3/2}$ and $u = T^{1/2}\lambda^{1/3}$. It is possible to do the inverse Laplace transform of the functions (3.11) and (3.12) explicitly and find the distribution of the excursion area. To do so we need the formula

$$\mathcal{L}^{-1}[\exp(-s\lambda^a); A] = \frac{as}{A^{a+1}}M_a(sA^{-a}) \tag{3.15}$$

where $M_{\frac{\beta}{2}}(z)$ is the well-known Wright function given by the following series:

$$M_{\frac{\beta}{2}}(z) = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-z)^k}{k!}\Gamma\left(\beta\frac{(k+1)}{2}\right)\sin\left(\beta\frac{(k+1)}{2}\right). \tag{3.16}$$

For the Dirichlet boundary condition the inverse Laplace transform for $T = 1$ gives

$$\begin{aligned} P^0(A) &= \sqrt{2\pi}\frac{2^{\frac{2}{3}}}{3}\sum_{k=1}^\infty c_i^0\partial_A\left(A^{-\frac{5}{3}}M\left(\frac{c_i^0}{2^{\frac{1}{3}}}A^{-\frac{2}{3}}, \frac{2}{3}\right)\right) \\ &= \frac{2\sqrt{6}}{A^{\frac{10}{3}}}\sum_{k=1}^\infty \mathrm{e}^{-\frac{2(c_i^0)^3}{27A^2}}\left(\frac{2\big(c_i^0\big)^3}{27}\right)^{2/3} U\left(-\frac{5}{6}, \frac{4}{3}, \frac{2\big(c_i^0\big)^3}{27A^2}\right). \end{aligned} \tag{3.17}$$

where $U(a, b, z)$ is the confluent hypergeometric function.

To calculate the moments of the area one can work in the Laplace space and then use the equality $\Gamma(n)\langle A^{-n}\rangle = \int_0^\infty P(\lambda, T)\lambda^{n-1}\mathrm{d}\lambda$. To do the calculations we need to first define the generalized Riemann function $\Lambda^\eta(s) = \sum_{i=1}^\infty \frac{1}{(c_i^\eta)^s}$ where c_i^η comes from equation (3.7). The above relations help us to calculate the moments explicitly as follows:

$$\langle A^n\rangle = \sqrt{2\pi}2^{\frac{1-n}{2}}\frac{n\Gamma\left(1+\frac{3(1-n)}{2}\right)}{\Gamma(2-n)}\Lambda^0\left(\frac{3(-n+1)}{2}\right). \tag{3.18}$$

It was shown in [14] that the moments after regularization are

$$\langle A^n\rangle = \sqrt{2\pi}2^{\frac{4-n}{2}}\frac{\Gamma(n+1)}{\Gamma\left(\frac{3n-1}{2}\right)}K_n \tag{3.19}$$

where K_n is by the following recursion relation:

$$K_n = \frac{3n-4}{4}K_{n-1} + \sum_{j=1}^{n-1}K_jK_{n-j}, \qquad s \geqslant 1, \tag{3.20}$$

the first few values are $K_0 = -\frac{1}{2}$, $K_1 = \frac{1}{8}$, $K_2 = \frac{5}{64}$ and $K_3 = \frac{15}{128}$. Then the first few moments of the excursion area are

$$\langle A^0\rangle = 1, \qquad \langle A^1\rangle = \frac{\sqrt{2\pi}}{4}, \qquad \langle A^2\rangle = \frac{5}{12}, \qquad \langle A^3\rangle = \frac{15\sqrt{2\pi}}{128}, \ldots. \tag{3.21}$$

There is also a nice relation between the Airy zeta function and K_n as follows:

$$\Lambda^0\left(\frac{3-3n}{2}\right) = -\frac{4}{3}\frac{\cos\left(\frac{3\pi n}{2}\right)}{\sin(\pi n)}K(n). \tag{3.22}$$

For example, we have the following limits:

$$\lim_{n\to 0} n\Lambda^0\left(\frac{3}{2}(n-1)\right) = \frac{2}{3\pi}, \qquad \Lambda^0(0) = \frac{1}{4}. \tag{3.23}$$

For the Neumann boundary condition we need another Laplace transform pair

$$\mathcal{L}^{-1}[\lambda^{-\alpha}\exp(-s\lambda^{-a}); A] = \frac{1}{A^{1-\alpha}}\phi(a, \alpha; -sA^a); -1 < a < 0, s > 0, \qquad 0 < \alpha < 1, \tag{3.24}$$

where $\phi(a, \alpha; z) = \sum_{k=0}^\infty \frac{z^k}{k!\Gamma(ak+\alpha)}$ is the generalized Wright function defined for the $a > -1$. Using the above formula the area distribution has the following form:

$$P^\infty(A) = \sqrt{\frac{\pi T}{2}}2^{1/3}\sum_{i=1}^\infty \frac{1}{c_i^\infty}\frac{\partial}{\partial A}\left(\frac{1}{A^{1/3}}\phi\left(\frac{-2}{3}, \frac{2}{3}; -2^{-1/3}c_i^\infty TA^{\frac{-2}{3}}\right)\right). \tag{3.25}$$

The moments of the area for $T = 1$ can be written as

$$\langle A^n\rangle = \frac{3\sqrt{\pi}}{2^{\frac{n+1}{2}}}\frac{\Gamma\left(\frac{1-3n}{2}\right)}{\Gamma(-n)}\Lambda^\infty\left(\frac{3-3n}{2}\right). \tag{3.26}$$

These moments are the same as the moments of the Brownian bridge and can be regularized in the same way [8]. Then the moments are

$$\langle A^n\rangle = \frac{\sqrt{\pi}2^{-n/2}\Gamma(1+n)}{\Gamma\left(\frac{1+3n}{2}\right)}D_n, n \geqslant 0, \tag{3.27}$$

where D_n is by the following recursion relation:

$$D_n = \frac{3n-2}{4}D_{n-1} - \frac{1}{2}\sum_{j=1}^{n-1}D_jD_{n-j}, \qquad s \geqslant 1. \tag{3.28}$$

The first few values are $D_0 = 1$, $D_1 = \frac{1}{4}$, $D_2 = \frac{7}{32}$ and $D_3 = \frac{21}{64}$. Then the first few moments of the area are

$$\langle A^0\rangle = 1, \qquad \langle A^1\rangle = \frac{1}{4}\sqrt{\frac{\pi}{2}}, \qquad \langle A^2\rangle = \frac{7}{60}, \qquad \langle A^3\rangle = \frac{21}{512}\sqrt{\frac{\pi}{2}}, \ldots. \tag{3.29}$$

It is not possible to find exact probability distribution for the generic η because for Robin boundary condition c_i^η is not independent of λ and since they are related by non-algebraic relation it is not possible to get c_i^η with respect to λ explicitly. However, one can follow the calculations in the level of Laplace space. The wavefunction has the following form:

$$\psi_i^\eta(x) = \frac{(2\lambda)^{1/6}\mathrm{Ai}\big((2\lambda)^{1/3}x - c_i^\eta\big)}{\sqrt{c_i^\eta \mathrm{Ai}^2\big(-c_i^\eta\big) + Ai'^2\big(-c_i^\eta\big)}}. \tag{3.30}$$

The Laplace transform of the distribution of the area is

$$P^\eta(\lambda, T) = \frac{(2\lambda)^{1/3}}{Z_E(\epsilon = 0)}\sum_{i=1}^{\infty}\left(\frac{\eta^2(2\lambda)^{2/3}}{4\pi^2}\right)\left(\frac{1}{1 + \left(\frac{\eta^2(2\lambda)^{2/3}}{4\pi^2}\right)c_i^\eta}\right)\mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^\eta T}. \tag{3.31}$$

To pursuit the calculation let us consider small ηs. One can write $c_i^\eta = c_i^0 + \delta c_i^\eta$ where δc_i^η is the small perturbation around the zeros of the Airy function. After expansion of (3.7) the perturbation is

$$\delta c_i^\eta \approx \frac{\eta(2\lambda)^{1/3}}{2\pi}. \tag{3.32}$$

For small η one can also expand Z_E as follows:

$$Z_E^\eta \approx \frac{\eta^2}{2^{3/2}\pi^{5/2}} + \mathcal{O}(\eta^4). \tag{3.33}$$

Then $P(\lambda, T)$ after expansion is

$$P^\eta(\lambda, T) \approx \sqrt{2\pi}\lambda T^{\frac{3}{2}}\sum_{i=1}^{\infty}\mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^0 T - \frac{\eta T}{2\pi}\lambda}. \tag{3.34}$$

The inverse Laplace transform of the function after using equation (3.15) is

$$P^\eta(A) \approx \sqrt{2\pi}\frac{2^{2/3}}{3}\sum_{k=1}^{\infty}c_i^0\partial_A\left(\left(A - \frac{\eta}{2\pi}\right)^{-\frac{5}{3}}M\left(\frac{c_i^0}{2^{\frac{1}{3}}}\left(A - \frac{\eta}{2\pi}\right)^{-\frac{2}{3}}, \frac{2}{3}\right)\right). \tag{3.35}$$

The moments of the area can be found by the same method as before by just replacing A in equation (3.19) by $A - \frac{\eta}{2\pi}$.

For the large ηs the same calculation can be done. The partition function of the elastic Brownian motion for large η is

$$Z_E^\eta \approx \sqrt{\frac{2}{\pi T}} - \frac{2\pi}{\eta} + \mathcal{O}\left(\frac{1}{\eta^2}\right). \tag{3.36}$$

The perturbation of c_i^η after expansion of equation (3.7) is

$$\delta c_i^\eta \approx \frac{-2\pi}{c_i^\infty\eta(2\lambda)^{1/3}}. \tag{3.37}$$

Unfortunately, since the perturbation of the energy is dependent on the energy level we are not able to find simple equation for the distribution of the area in this case.

3.2. The area under the elastic Brownian meander

The definition of the elastic Brownian meander is similar to the elastic Brownian excursion, the only difference is for the elastic Brownian meander we do not need to force the process to come back to the starting point. In this case the partition function is

$$Z_M^\eta = \int_0^\infty \mathrm{d}b \langle b|\, \mathrm{e}^{-H_0 T}|\epsilon\rangle. \tag{3.38}$$

One can show that

$$\langle b|\mathrm{e}^{-H_0 T}|\epsilon\rangle = \sqrt{\frac{1}{2\pi T}}\left(\mathrm{e}^{-\frac{1}{2}\frac{(\epsilon-b)^2}{T}} + \mathrm{e}^{-\frac{1}{2}\frac{(\epsilon+b)^2}{T}}\right) + \frac{2\pi}{\eta}\mathrm{e}^{\frac{2\pi^2}{\eta^2}T - \frac{2\pi(\epsilon+b)}{\eta}} Erfc\left[\left(\frac{2\pi^2 T}{\eta^2}\right)^{1/2} - \frac{\pi(\epsilon+b)}{\eta}\left(\frac{2T\pi^2}{\eta^2}\right)^{-1/2}\right]. \tag{3.39}$$

For small η one can expand the error function up to the second order

$$\langle b|\mathrm{e}^{-H_0 T}|\epsilon\rangle \approx \frac{1}{\sqrt{2\pi T}}\left(\mathrm{e}^{-\frac{1}{2}\frac{(\epsilon-b)^2}{T}} + \mathrm{e}^{-\frac{1}{2}\frac{(\epsilon+b)^2}{T}}\right) - \frac{2}{\sqrt{2\pi T}}\frac{\mathrm{e}^{-\frac{1}{2}\frac{(\epsilon+b)^2}{T}}}{\left(1-\eta\frac{\epsilon+b}{2\pi T}\right)}. \tag{3.40}$$

Taking the first-order correction with respect to η and integrating over b gives

$$Z_M^\eta \approx Erf\left(\frac{\epsilon}{\sqrt{2T}}\right) - \frac{\eta}{\pi\sqrt{2\pi T}}\,\mathrm{e}^{-\frac{\epsilon^2}{2T}}. \tag{3.41}$$

For $\eta \to \infty$ it is easy to get

$$Z_M^\infty \approx 1. \tag{3.42}$$

Similar to the calculation that we did in the elastic Brownian excursion case one can write the Laplace transform of the distribution of area as

$$P^\eta(\lambda, T) = \frac{1}{Z_M^\eta}\int_0^\infty \mathrm{d}b\langle b|\,\mathrm{e}^{-H_1 T}|\epsilon\rangle. \tag{3.43}$$

Using the wavefunction (3.30) one can get

$$P^\eta(\lambda, T) = \frac{1}{Z_M^\eta}\sum_{i=1}^\infty \frac{\mathrm{Ai}\left((2\lambda)^{1/3}\epsilon - c_i^\eta\right)\int_{-c_i^\eta}^\infty \mathrm{Ai}(y)\mathrm{d}y}{c_i^\eta \mathrm{Ai}^2\left(-c_i^\eta\right) + \mathrm{Ai}'^2\left(-c_i^\eta\right)}\mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^\eta T}. \tag{3.44}$$

After expansion of the function with respect to ϵ and η the first correction appears in the spectrum as follows:

$$P^\eta(\lambda, T) = \sqrt{\pi}2^{-1/6}(\lambda T^{3/2})^{1/3}\sum_{i=1}^\infty B\left(c_i^0\right)\mathrm{e}^{-2^{-1/3}\lambda^{2/3}c_i^\eta T}. \tag{3.45}$$

where $B(c_i^0) = \frac{\int_{-c_i^0}^\infty \mathrm{Ai}(y)\mathrm{d}y}{\mathrm{Ai}'(-c_i^0)}$ and $c_i^\eta = c_i^0 + \delta c_i^\eta$. The distribution of the area after inverse Laplace transform is

$$P^\eta(A, T) = \sqrt{\pi}2^{-1/6}T^{1/2}\sum_{i=1}^\infty B\left(c_i^0\right)\frac{\partial}{\partial A} \times \left(\frac{1}{\left(A-\frac{\eta}{2\pi}\right)^{1/3}}\phi\left(\frac{-2}{3},\frac{2}{3};-2^{-1/3}c_i^\eta T\left(A-\frac{\eta}{2\pi}\right)^{\frac{-2}{3}}\right)\right). \tag{3.46}$$

The continuation of calculation is now straightforward, we just need to use the well-known results for the Brownian meander. The moments of the area for Brownian meander, i.e. $\eta = 0$, for $T = 1$ are

$$\langle A^n \rangle = \sqrt{\pi} 2^{-n/2} \frac{\Gamma(n+1)}{\Gamma\left(\frac{3n+1}{2}\right)} Q_n. \tag{3.47}$$

Q_n satisfies the following recursion relations:

$$\begin{aligned} Q_n &= \beta_n - \sum_{j=1}^{n} \alpha_j Q_{n-j}, \\ \beta_n &= \alpha_n + \frac{3}{4}(2n-1)\beta_{n-1}, \\ \alpha_n &= \frac{6^{-2n}}{\Gamma(n+1)} \frac{\Gamma(3n+1/2)}{\Gamma(n+1/2)}. \end{aligned} \tag{3.48}$$

The first few values are $Q_0 = 1$, $Q_1 = \frac{3}{4}$, $Q_2 = \frac{59}{32}$ and $Q_3 = \frac{465}{64}$. Then the first few values of the moments of the area are

$$\langle A^0 \rangle = 1, \qquad \langle A^1 \rangle = \frac{3}{4}\sqrt{\frac{\pi}{2}}, \qquad \langle A^2 \rangle = \frac{59}{60}, \qquad \langle A^3 \rangle = \frac{465}{512}\sqrt{\frac{\pi}{2}}, \ldots. \tag{3.49}$$

To get the results for the small η one needs to replace A with $A - \frac{\eta}{2\pi}$ in (3.49).

The next interesting case is the Neumann boundary with $\eta \to \infty$. The Laplace transform of the distribution of the area after a little algebra is

$$P^{\infty}(\lambda, T) = \sum_{i=1}^{\infty} \kappa_i \, \mathrm{e}^{-2^{-1/3}\lambda^{2/3} c_i^{\infty} T}, \tag{3.50}$$

where $\kappa_i = \frac{AI(c_i^{\infty})}{c_i^{\infty} Ai(c_i^{\infty})}$ with $AI(z) = \int_z^{\infty} \mathrm{Ai}(y)\mathrm{d}y$. This distribution is exactly the same as the distribution of the area of the Brownian motion, i.e. $\int_0^T |B_t| \mathrm{d}t$. This is not surprising because the absolute value of the Brownian motion is like considering the area in the presence of totally reflecting boundary condition. Using the inverse Laplace transform one can get

$$P^{\infty}(A, T) = \frac{2^{-1/3} T}{A^{5/3}} \sum_{i=1}^{\infty} \kappa_i c_i^{\infty} M_{\frac{2}{3}} \left(\frac{2^{-1/3} c_i^{\infty} T}{A^{2/3}} \right) \tag{3.51}$$

Using the well-known results for the area of the absolute value of the Brownian motion [8] the moments of the area can be written as

$$\langle A^n \rangle = \frac{2^{-n/2} \Gamma(1+n)}{\Gamma\left(\frac{3n+2}{2}\right)} L_n, \tag{3.52}$$

where L_n satisfies the following recursion relation:

$$L_n = \beta_n + \sum_{i=1}^{n} \frac{6j+1}{6j-1} \alpha_j L_{n-j}, \tag{3.53}$$

and α_j and β_j are the same as in equation (3.48). The first few values are $L_0 = 1$, $L_1 = 1$, $L_2 = \frac{9}{4}$ and $L_3 = \frac{263}{32}$. Then the first few values of the moments of the area are

$$\langle A^0 \rangle = 1, \qquad \langle A^1 \rangle = \frac{2}{3}\sqrt{\frac{2}{\pi}}, \qquad \langle A^2 \rangle = \frac{3}{8}, \qquad \langle A^3 \rangle = \frac{263}{630}\sqrt{\frac{2}{\pi}}, \ldots. \tag{3.54}$$

3.3. Some applications

Different applications of the area distributions of the Brownian motion in computer science, graph theory, fluctuating one-dimensional interfaces and localization in electronic systems were already discussed at length in many papers, see [16, 17] and references therein. The Airy distribution function and its derivative appear extensively in all applications. In this subsection, we will summarize some immediate simple applications of our extended Airy distribution. We will discuss the distribution of the average distance of a particle from a disorder with point interaction in three dimensions and the distribution of the average relative height distribution of interacting interfaces in two dimensions. We will also discuss one more elastic Brownian functional distribution related to vicious walkers interacting with the boundary.

The first immediate application comes from the equality of the norm of the three-dimensional Brownian motion (called three-dimensional Bessel process) and Brownian motion on the half line [1, 10]. This is very easy to show by considering the radial part of the Fokker–Planck equation of the three-dimensional Brownian motion. Now consider a Brownian motion moving in three dimensions in the presence of the disorder at the origin, one can map the system to the problem of one particle on the half line. The most generic point interaction between disorder and particle comes from the self-adjoint extension of the Hamiltonian of the free particle in punctured three-dimensional space [10] which is equal to the free particle on the half line with the boundary condition that we discussed in section 2. Then it is easy to see that the area distribution that we calculated is just the average distance distribution in the period T between the disorder and the free particle with the generic point interaction.

Another simple application, which is in the close connection with the previous example, is the interacting interfaces. A path of Brownian motion in the x–t space is just like an interface with zero roughness exponent. One can also look at this interface as an ensemble of growth models such as the Edwards–Wilkinson model. Consider now two non-crossing interfaces in the region $[0, L]$ with the similar boundary conditions. This problem is equivalent to the problem of two free particles on the real line. Consider, like the previous example, point interactions between the particles. The interaction between the particles is equivalent to the interaction between the interfaces. Then the area distribution that we calculated for the elastic Brownian excursion is just the distribution of the average distance of the interfaces in the interval $[0, L]$. One can also relax the boundary condition in one of the end points and consider different boundary conditions for the different interfaces and use the results corresponding to elastic Brownian meander area.

Since the elastic Brownian motion is the generalization of the Brownian motion in the presence of the boundary we believe that all the possible applications should deal with the boundary interaction or point interaction between two particles. In this paper, we just calculated one possible functional of the elastic Brownian motion, the area. However, there are many other functionals that can have interesting physical applications in the study of the interacting non-crossing walkers or interacting interfaces. We will discuss some of these functionals in the conclusion of the paper and give here one more example with more detail.

Consider the problem of p non-crossing walkers, which has application in describing domain walls of elementary topological excitations in the commensurate adsorbed phases close to the commensurate–incommensurate transition [18], in the presence of the boundary. One interesting quantity is the maximal height distribution of the top walker that was already calculated exactly for the vicious walkers in [19]. Walkers are vicious if they annihilate each other when they meet. For simplicity we will discuss the simplest case $p = 1$. Consider H as the maximal height of the walker in $[0, 1]$ then one can define the cumulative distribution as

$P(M) = \text{Prob}[H \leqslant M]$. We define $N(\epsilon, M)$ as the probability that the walker do an excursion in the period [0, 1] starting at ϵ and coming back to the same point staying within the interval $[0, M]$. The cut-off ϵ is just necessary for the Dirichlet boundary condition as we discussed before. Since for this case the result is already known [20] we will put $\epsilon = 0$ hereafter. Then it is easy to show that $N(0, M) = \sum_E |\psi_E|^2 \mathrm{e}^{-E}$. The wavefunction is as (2.5), i.e. $\psi_E = \sqrt{\frac{2}{\pi}} \cos(kx + \delta_k)$, with $E = \frac{k^2}{2}$ and k is the solution to the equation $\cot(kM) = \frac{2\pi}{\eta k}$. Finally, one can write the cumulative distribution as

$$P(M) = \frac{1}{Z^\eta} \sum_k \frac{1}{\left(1 + \left(\frac{2\pi}{\eta k}\right)^2\right)\left(\frac{M}{2} - \frac{1}{4k}\sin(2kM)\right)} \mathrm{e}^{-k^2/2}, \tag{3.55}$$

where Z^η comes from equation (3.3). For small η and small M with $M < \eta$ one can simplify the equation as $P(M) \approx \frac{\sqrt{2\pi}}{M} k_0^2 \, \mathrm{e}^{-k_0^2/2}$, where $k_0 = \frac{\sqrt{3 - \frac{6\pi M}{\eta}}}{M}$.

Generalization of the above results to arbitrary p is straightforward, however, it is rather cumbersome. From the perspective of our study in this paper one can generalize this problem in two directions: the first possibility is considering non-vicious walkers, interacting domain-walls, with a Dirichlet boundary condition on the wall. The second possibility is considering vicious walkers with non-trivial interaction with the boundary.

4. The area of the Brownian bridge on the line with a point defect

In this section, we will summarize that some results come from the area calculation for the Brownian bridge on the line with a point defect. The quantum-mechanical counterpart was discussed in length in the literature and it is called the free particle on a line with a hole [15]. The functional integral for this problem was discussed in [13] and it is based on the different local times of the particle in the two sides of the origin. The most general boundary condition that respects the time reversal symmetry for the defect on the origin, after using self-adjoint extension theory, is

$$\begin{pmatrix} \psi'_+(0) \\ \psi_+(0) \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ -\delta & -\gamma \end{pmatrix} \begin{pmatrix} \psi'_-(0) \\ \psi_-(0) \end{pmatrix}, \tag{4.1}$$

with a constraint $\alpha\gamma - \beta\delta = 1$. For simplicity we will consider some special cases. It is easy to see that for $\delta \to \infty$ the boundary condition decouples and we will have two decoupled half lines and so the area distribution is as the previous section.

The next simple case is $\alpha = \gamma = -1$ and $\delta = 0$ which is equal to the free particle on the line with the following delta function potential:

$$V(x) = -\frac{\beta}{2}\delta(x). \tag{4.2}$$

To avoid the bound state we consider non-positive β. To calculate the area of this kind of Brownian bridge one can use the method of the previous section. Interestingly, the results are very similar to the previous section. The energy of the particle is $E_k = \frac{1}{2}k^2$ and the wavefunctions are

$$\psi_k(x) = \sqrt{\frac{1}{\pi}} \cos(k|x| + \delta_k), \tag{4.3}$$

where $\tan(\delta_k) = \frac{\beta}{2k}$. Then Z_E^β is the same as (3.3) after replacing η with $\frac{4\pi}{\beta}$. The wavefunctions in the presence of the potential $\lambda|x|$ can have different parities, they are

$$\psi_i^{\beta}(x) = \frac{2^{-1/2}(2\lambda)^{1/6}\mathrm{Ai}((2\lambda)^{1/3}|x| - c_{\beta i})}{\sqrt{c_{\beta i}\mathrm{Ai}^2(-c_{\beta i}) + \mathrm{Ai}'^2(-c_{\beta i})}}, \qquad \text{even parity,} \tag{4.4}$$

$$\psi_i(x) = \mathrm{sgn}(x)2^{-1/2}(2\lambda)^{1/6}\frac{\mathrm{Ai}((2\lambda)^{1/3}|x| - c_{-\infty i})}{\mathrm{Ai}'(-c_{-\infty i})}, \qquad \text{odd parity,} \tag{4.5}$$

where c_β is the same as c^η after replacing η with $\frac{4\pi}{\beta}$. Odd wavefunctions do not contribute in the distribution of the area. The result for the even part is the same as the result for the elastic Brownian excursion. It is easy to see that $\beta = 0$ is like the free Brownian motion and so the distribution of the area is like Brownian bridge or like elastic Brownian excursion with Neumann boundary condition, i.e. (3.25). $\beta = -\infty$ is like the Dirichlet boundary condition and the distribution of the area is (3.17) after considering $\eta = 0$.

The other simple boundary condition comes from $\alpha = \gamma = -1$, $\beta = 0$ and $\delta \neq 0$, in the other words

$$\psi_+'(0) - \psi_-'(0) = 0, \tag{4.6}$$

$$\psi_+(0) - \psi_-(0) = -\delta\psi_-'(0). \tag{4.7}$$

The energy of the particle is $E_k = \frac{1}{2}k^2$ and the wavefunctions are

$$\psi_k(x) = \mathrm{sgn}(x)\sqrt{\frac{1}{\pi}}\cos(k|x| + \delta_k), \tag{4.8}$$

where $\tan(\delta_k) = \frac{2}{\delta k}$. Then Z_E^δ is the same as (3.3) after replacing η with $\pi\delta$.

The wavefunctions in the presence of the potential $\lambda|x|$ can have different parities, they are

$$\psi_i^{\delta}(x) = \mathrm{sgn}(x)\frac{2^{-1/2}(2\lambda)^{1/6}\mathrm{Ai}\big((2\lambda)^{1/3}|x| - c_i^{\pi\delta}\big)}{\sqrt{c_i^{\pi\delta}\mathrm{Ai}^2\big(-c_i^{\pi\delta}\big) + \mathrm{Ai}'^2\big(-c_i^{\pi\delta}\big)}}, \qquad \text{odd parity,} \tag{4.9}$$

$$\psi_i(x) = (2\lambda)^{1/6}\frac{\mathrm{Ai}\big((2\lambda)^{1/3}|x| - c_i^{\infty}\big)}{\sqrt{2c_i^{\infty}}\mathrm{Ai}\big(-c_i^{\infty}\big)}, \qquad \text{even parity.} \tag{4.10}$$

The above calculation shows that in this case the distribution of the area can be calculated by adding two terms. In this case, the even parity has contribution for the distribution of the area which is equal to the Neumann boundary condition. Replacing η with $\pi\delta$ in the formula of elastic Brownian excursion will give the contribution of the odd part. The case of $\delta \to \infty$ is equal to the Neumann boundary condition that separates the system into the two regions, positive and negative parts of the real line. Of course having two solutions is an indicator of degeneracy resulting from the parity invariancy of the system in this limit. $\delta = 0$ is the free particle case and one can see that only the even parity has contribution in the area distribution.

5. Conclusion and discussion

In this paper, we found the area distribution of the elastic Brownian motion in some limits. Our method was based on the equality of this process with the self-adjoint Hamiltonian of the quantum particle on the half line. The corresponding Hamiltonian for the area distribution is just the Hamiltonian with linear potential. The eigenvalues of this Hamiltonian after self-adjoint extension satisfy a transcendental equation and so for the generic case the distribution of the area is not available, however, in some limits the calculation is tractable. We found perturbatively the area distribution of the Brownian excursion and the Brownian meander in the presence of the weekly reflecting barrier. By using self-adjoint extension we found a unified method to classify different possible area distribution for the Brownian motion in the presence of the boundary. Some possible applications in diffusion phenomena in the presence of disorder and interacting interfaces were also discussed.

We did similar calculations for the Brownian motion on the pointed line. The self-adjoint Hamiltonian in this case has three independent parameters and the eigenvalues of the Hamiltonian satisfy the same transcendental equation in some interesting limits. Similar calculations can be useful in describing different distributions of diffusing particles in the presence of point disorder.

Plenty of questions remain to be answered in the study of the elastic Brownian motion by using quantum-mechanical techniques. Some of them are as follows: the case of the maximal height of p non-intersecting Brownian excursions and Brownian bridges is also interesting to be calculated [19], the possible connection of this study to the interacting domain walls of elementary topological excitations in the commensurate adsorbed phases close to the commensurate–incommensurate transition can be interesting. The other example is the distribution of the time to reach the maximum [21]. Unfortunately, the eigenvalue equations for the above cases are transcendental as we faced for one example in the end of section 3 and so it is impossible to get a closed formula for the distributions, however, the exact calculations in some limits may be possible. The other example is the distribution of the time spent by the particle on the positive side of the origin out of the total time t. This distribution was first calculated by Lévy in the case of the Brownian motion [22]. For the pointed line the equations are again transcendental and need to be solved by numerical calculation. We mostly focused on the distributions in one dimension, however, one can also try to calculate the different distributions like the algebraic area distribution, winding number distribution in the pointed two-dimensional space, the number of defects could be finite or infinite[2]. We believe that the method of self-adjoint extension in quantum mechanics can be useful to calculate such kind of distributions. It is also interesting to check our results with the numerical calculations.

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